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Stochastically driven dynamical systems

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Initially, this study arose from questions concerning the scattering of tsunamis as they propagate over the irregular topography of the deep waters of the ocean. The mathematical problem to which this led is pertinent to many other phenomena, however, and we direct the analysis, here, to the propagation of gravity waves over an irregular bottom topography and to the lateral oscillations of an elastic string whose ends undergo random longitudinal displacements. Several facets of the mathematical problem are rather fascinating but the results do suggest that scattering is not the most important part of the tsunami propagation.

1. Introduction

When a tsunami travels from the Aleutian Islands towards Hawaii, most of the wave energy lies in a spectral range which is characterized by a large horizontal scale compared to the depth of the ocean. Accordingly, the dynamics of the propagation could be sensitive to certain features of the bottom topography. Only waves whose lateral scales are comparable to that of the bottom topography could be significantly reflected, of course, but much of the energy is associated with waves which are not excluded by this observation. In particular, if one pretended that the depth distribution, $H = H_0(1+\epsilon\cos 2k_0x)$, was a suitable approximation for the study of such reflexions, one would use $\epsilon \simeq 1/50$ and $\lambda = 2\pi/k_0 = O(100 \text{ miles})$. With such figures, one would predict that most of the energy in the spectral band, $|k-k_0| < \frac{1}{2}\epsilon k_0$, would never reach the 'target'. There is no evidence that suggests that such 'blocking' occurs and one would like to be able to calculate more suitable estimates of the reflexion of waves travelling over an irregular bottom topography. The analysis of Carrier (1966) does not suffice for this purpose because the strength of the reflexion (even though it may be numerically important) is transcendentally small in the expansion parameter of that analysis; and, since that expansion is asymptotic in character, it cannot account for transcendentally small effects.

In this paper, we will develop a scheme whereby the strength of such reflexions can be estimated. In this scheme, we will treat the phenomenon as though the bottom topography were one-dimensional and as though it were one realization of a family of bottom topographies whose properties are known stochastically. We adopt this point of view not only because no precise description of the topography is available, but also because the appropriate one-dimensional 'sections' of that topography will vary from one tsunami to another in accord with their different locations. Thus, in a very real sense, we are studying an ensemble of problems characterized by an ensemble of depth distributions. The palatability of the decision to treat the problem as a stochastic one is also enhanced by the observation that the deterministic problem with a realistic depth distribution would be very difficult to solve.

The surface slopes associated with the tsunami are so small during the propagation over the deep water that, even with the long propagation path, non-linear effects are entirely negligible; furthermore, the consequences of dispersion are already understood (Carrier 1966). Thus, an analysis of the propagation of a monochromatic wave in a region of irregular depth using linear shallow water theory will meet our needs. In such a theory the wave height $\eta(x')e^{i\omega t}$ obeys the equation

$$(F(x')\eta_{x'})_{x'} - g^{-1}\eta_{tt} = 0, \tag{1.1}$$

where $F(x')$ is the position-dependent depth. When one defines

$$x(x') = C\int_0^{x'} dx''/F(x'') \tag{1.2}$$

and

$$u(x) = \eta(x'),$$

where $C^2 = \omega^2 F_0/g$, F_0 is the average depth, and $F = F_0(1+\epsilon f(x))$, (1.1) becomes

$$u_{xx} + [1+\epsilon f(x)]u = 0. \tag{1.3}$$

A very convenient geometry for our purposes is that for which

$$f(x) = \begin{cases} 0 & \text{in} \quad x < 0, \\ \text{stochastically characterized} & \text{in} \quad 0 < x < L, \\ 0 & \text{in} \quad x > L, \end{cases}$$

and in which the incident wave travels in the negative x direction.

In this geometry we required that $u(x) = e^{ix}$ in $x < 0$ for every realization of $f(x)$. This condition uniquely implies the solution elsewhere, since u and u' must be continuous at $x = 0$. In particular,

$$u = Ie^{ix} + \mathscr{R}e^{-ix} \quad \text{in} \quad x > L$$

and, since Ie^{ix} is the incident wave and $\mathscr{R}e^{-ix}$ is the reflected wave, $\mathscr{R}/I$ is the reflexion coefficient. The conservation of energy implies

$$|I|^2 - |\mathscr{R}|^2 = 1, \tag{1.4}$$

so we need only information about $|\mathscr{R}|^2 + |I|^2$ to infer information about the reflexion produced by a domain of length L. In particular, since

$$u(L) = Ie^{iL} + \mathscr{R}e^{-iL} \quad \text{and} \quad u'(L) = i(Ie^{iL} - \mathscr{R}e^{-iL})$$

we see that

$$|I|^2 + |\mathscr{R}|^2 = \tfrac{1}{2}\{|u(L)|^2 + |u'(L)|^2\}. \tag{1.5}$$

Furthermore, since the conditions $u(0) = 1$ and $u'(0) = i$ imply a unique solution in $x > 0$ (for each sample function, $f(x)$), our initial-value problem is given by

$$u''(x) + [1+\epsilon f(x)]u(x) = 0 \quad \text{in} \quad x > 0 \tag{1.6}$$

with $u(0) = 1$, $u'(0) = i$.

Our primary objective is to find

$$P = \langle |u(L)|^2 + |u'(L)|^2 \rangle \quad \text{for} \quad L > 0, \tag{1.7}$$

where $\langle\ \rangle$ denotes the ensemble average over the set of solutions $u(x)$. It is unlikely that one can easily obtain a characterization of the functions $1+\epsilon f$ from which one can deduce more than $\langle f(x)f(y)\rangle$ and ϵ. Accordingly, we will try to devise a procedure for estimating P which utilizes only information about the 'average size', ϵ, of the topographical variations and the auto-correlation function, R, of those variations. We will also admit only stationary functions $f(x)$; that is,

$$\langle f(x)f(y)\rangle = R \equiv R(x-y). \tag{1.8}$$

Furthermore, we normalize R according to the rule

$$\int_{-\infty}^{\infty} R(x)\,dx = 1 \tag{1.9}$$

and this ensures a unique meaning for ϵ.

2. The vibrating string

It is well known (Lubkin & Stoker 1943) that, when the distance, L, between the ends of an elastic string (whose fundamental mode has frequency ω_0 when $L = L_0$) is required to be $L = L_0(1+\epsilon\cos 2\omega_0 t)$, the lateral oscillations of the string will grow in amplitude until the average (over time) of the tension is significantly affected by the stretching which accompanies the lateral motion. At such amplitudes, the motion ceases to increase in amplitude. It is very interesting to study the corresponding phenomenon when the separation of the ends of the string is given by

$$L = L_0 + H(t),$$

where $H(t)$ is known only in a statistical sense. Of particular interest is the comparison of the rates of growth of the lateral displacement and the contrast in the character of the consequences of the tensile change implied by large displacements.

The behaviour of an elastic string, when the non-linear effects of large displacements are significant, has been studied extensively (e.g. McLachlan 1950; Carrier 1949). A quick, oversimplified derivation of an appropriate mathematical model notes that, because longitudinal waves travel so fast, one can approximate the tension in the string, $T_1(x,t)$, by $T_1(x,t) \simeq T(t)$. The quantity T is given by

$$T \simeq T_0 + EA\left[H(t)/L_0 + \frac{1}{2}\int_0^{L_0} u_x^2\,dx\right]. \tag{2.1}$$

Here T_0 is the tension when the string is straight and of length L_0; E is the Young's modulus of the material, A is its cross-sectional area, and $H(t) = (L_0T_0/EA)\epsilon f(t)$ is the longitudinal displacement of the right-hand end of the string (see figure 1) at time t. Thus, EAH/L_0 is the additional tension associated with the contribution of $H(t)$ to the stretching. When x, $u(x,t)$ are the co-ordinates of the position of the string at time t, the integral term of (2.1) is the contribution to the tension which accompanies the stretching due to the bowing of the string.

Note that $u(x,t)$ is *not* the lateral displacement at time t of the particle whose rest position is at x, but it *is* a good enough approximation to that displacement to be used as though it really were. With that approximation, the conservation of momentum in the lateral direction is

$$(Tu_x)_x = \rho A u_{tt}. \tag{2.2}$$

Consistent with this equation, we shall limit our study to oscillations in which

$$u(x,t) = u(t)\sin(\pi x/L),$$

whereupon $T/\rho A$ can be written as

$$T/\rho A = t_0^2[1+\epsilon f(t)+u^2], \tag{2.3}$$

so that, when t and $u(t)$ are measured in appropriate units,

$$u''+[1+\epsilon f(t)]u+u^3 = 0. \tag{2.4}$$

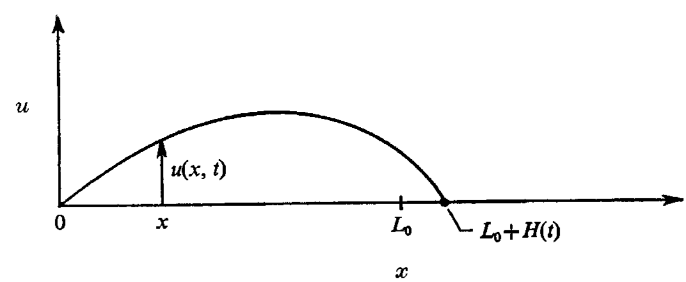

FIGURE 1. Sketch of string indicating the notation of §2.

We are interested in the behaviour of $u(t)$ when, for example,

$$u(0) = u_0 \ll 1, \quad u'(0) = 0. \tag{2.5}$$

When u is so small that the term u^3 is unimportant, the problem degenerates to a linear one and we shall treat that first.

3. Analysis of the stochastic problem

The problem implied by the foregoing discussion is: How can one describe statistical properties of the ensemble of solutions $u(t)$ of the initial-value problem

$$u''+[1+\epsilon f(t)]u = 0 \quad \text{in} \quad t>0 \tag{3.1}$$

with

$$u(0) = 1, \quad u'(0) = 0, \tag{3.2}$$

when we know nothing about the ensemble of functions $f(t)$ except that

$$\langle f(t)f(\tau)\rangle = R(t-\tau). \tag{3.3}$$

Generally speaking, we can do very little that is useful under these circumstances unless we adopt some hypotheses which *cannot* be true with full generality but which are rigorously correct in very special circumstances and which are excellent approximations to the truth in a wide variety of circumstances. The need for these hypotheses is most easily displayed by noting that (3.1), (3.2) imply that

$$u(t) = \cos t + \epsilon \int_0^t \sin(t-t')f(t')\,u(t')\,dt' \tag{3.4}$$

and, therefore, that

$$\begin{aligned} u(t)\,u(\tau) = {} & \cos t \cos \tau + \epsilon \cos t \int_0^\tau \sin(\tau-\tau')f(\tau')\,u(\tau')\,d\tau' \\ & + \epsilon \cos \tau \int_0^t \sin(t-t')f(t')\,u(t')\,dt' \\ & + \epsilon^2 \int_0^\tau \int_0^t \sin(t-t')\sin(\tau-\tau')f(t')f(\tau')\,u(t')\,u(\tau')\,dt'\,d\tau'. \end{aligned} \tag{3.5}$$

We take the ensemble average of each term in this equation, noting, e.g., that, whenever A is deterministic (and is therefore a common factor of each term being averaged) but g and h are random quantities,

$$\left\langle \int_0^t A(t,t')g(t')h(t)\,dt' \right\rangle = \int_0^t A(t,t')\,\langle g(t')h(t)\rangle\,dt'.$$

We obtain an equation with three different unknown quantities, i.e.

$$\langle u(t)\,u(\tau)\rangle, \quad \langle f(t)\,u(t)\rangle, \quad \text{and} \quad \langle f(t)f(\tau)\,u(t)\,u(\tau)\rangle.$$

The first of these, if known, would describe very well those properties of u we would like to know and could reasonably hope to estimate with our limited knowledge about f.

When $\epsilon \ll 1$, $u(t)$ is well approximated over any time interval $\Delta t \ll 1/\epsilon$ by $A\cos(t-\phi)$, where A and ϕ are constants. That is, only small deviations in phase and amplitude from a purely trigonometric behaviour can be displayed by a solution of (3.1) over such a time interval. Thus, if we confine our justification arguments to cases where knowledge of the value of f at t implies knowledge about $f(\tau)$ only when $|t-\tau| \ll 1/\epsilon$, then we can hope that $\langle f(t)\,u(t)\rangle$ is very small and can be ignored.

Similarly, we can define

$$\rho(t,\tau) = u(t)\,u(\tau) - \langle u(t)\,u(\tau)\rangle$$

and

$$p(t,\tau) = f(t)f(\tau) - \langle f(t)f(\tau)\rangle.$$

It follows that

$$\langle u(t)\,u(\tau)f(t)f(\tau)\rangle = \langle u(t)\,u(\tau)\rangle\,\langle f(t)f(\tau)\rangle + \langle \rho(t,\tau)\,p(t,\tau)\rangle. \tag{3.6}$$

Again, however, we expect $\rho(t,\tau)$ to be small when $p(t,\tau)$ is significant and we can ignore the final term in (3.6).

If we adopt the approximations (hypotheses)

$$\langle f(t)\,u(t)\rangle = 0 \tag{3.7}$$

and

$$\langle f(t)f(\tau)\,u(t)\,u(\tau)\rangle = \langle u(t)\,u(\tau)\rangle R(t-\tau), \tag{3.8}$$

the ensemble average of (3.5) contains only one unknown quantity: namely $\langle u(t)\,u(\tau)\rangle \equiv \psi(t,\tau)$ and the integral equation for $\psi(t,\tau)$ is

$$\psi(t,\tau) = \cos t \cos \tau + \epsilon^2 \int_0^t \int_0^\tau \sin(t-t')\sin(\tau-\tau')R(t'-\tau')\,\psi(t',\tau')\,dt'\,d\tau'. \tag{3.9}$$

In § 7 we will establish further the validity of these conjectures. In this section we content ourselves with the solution of the problem and examination of its implications.

For most of the ensembles of functions $f(t)$ which we will consider, we adopt the normalization

$$\int_{-\infty}^{\infty} R(x)\,dx = 1.$$

No generality is lost by this choice since ϵ can be chosen to provide consistency with any given situation (i.e. any given set of functions $f(t)$).

It is convenient, first, to treat the problem for which $R(t)$ is 'narrow'; that is, for which

$$\int_{-\infty}^{\infty} x^2 F(x)\,dx \ll 1.$$

For that case $R(t'-\tau')$ can be replaced in (3.9) by

$$R(t'-\tau') \simeq \delta(t'-\tau').$$

Then, with $t = \tau$, and with $\psi(t,t) \equiv \phi(t)$

$$\phi(t) = \cos^2 t + \epsilon^2 \int_0^t \sin^2(t-t')\,\phi(t')\,dt'. \tag{3.10}$$

The Laplace transform of $\phi(t)$ is

$$\overline{\phi}(S) = \frac{S^2+2}{S(S^2+4)-2\epsilon^2} \tag{3.11}$$

and

$$\phi(t) \simeq \tfrac{1}{2}(e^{\frac{1}{2}\epsilon^2 t} - e^{-\frac{1}{4}\epsilon^2 t}) + \cos^2 t\, e^{-\frac{1}{4}\epsilon^2 t}. \tag{3.12}$$

Note that the growth rate of the oscillation depends only on the magnitude, ϵ, of the random coefficient and note that only after a time of order ϵ^{-2} has the oscillation 'forgotten' its initial conditions.

For future reference, it is useful to note that the substitution of $\psi(t,t) = \phi(t)$ into the right-hand side of (3.9) permits the calculation of $\psi(t,\tau)$.
The asymptotic form of the result for large t and τ can be written

$$\psi(r,\tau) \sim e^{\frac{1}{4}\epsilon^2(t+\tau)}\, e^{-\frac{1}{4}\epsilon^2|t-\tau|} \cos(t-\tau). \tag{3.13}$$

In the string context, we see (directly from (3.10) or by inference from (3.11)) that the lateral oscillations will continue to grow like $e^{\frac{1}{2}\epsilon^2 t}$ for as long as (3.1) is a valid model of the dynamics, and that (statistically speaking) the phase of the oscillation wanders significantly only over time intervals comparable to the e-folding time of the amplitude growth.

When R is not narrow enough to permit the foregoing approximation, we proceed as follows.

We note first that the size of $\phi(t) = \psi(t,t)$ will not be altered if we choose an earlier initial time, say $t_0 < 0$, provided that we choose an appropriately smaller initial value, $u(t_0)$. The useful extreme of this is to take

$$u(-\infty) = 0$$

as the initial condition and to normalize the solution of the homogeneous problem to which this leads by specifying that $\phi(0) = 1$. When this is done, there will be no direct display of the way in which an initial condition at $t = 0$ would have constrained the phase at later times but we can easily infer this information when we write $\psi(t, \tau)$ as the product of a function of $t+\tau$ and of $t-\tau$. The latter factor, as we shall see, gives a clear picture of this constraint on the phase of $u(t)$.

When we put the initial time at $-\infty$, (3.9) becomes

$$\psi(t,\tau) = \epsilon^2 \int_{-\infty}^{t} \int_{-\infty}^{\tau} \sin(t-t')\sin(\tau-\tau')R(t'-\tau')\psi(t',\tau')\,dt'\,d\tau'. \quad (3.14)$$

It is convenient to define new variables

$$\alpha = t+\tau, \quad \beta = t-\tau, \quad \psi(t,\tau) = \chi(\alpha,\beta), \quad (3.15)$$

and (3.14) can then be written

$$\chi(\alpha,\beta) = \tfrac{1}{4}\epsilon^2 \int_{-\infty}^{\infty} d\beta' \int_{-\infty}^{\alpha-|\beta-\beta'|} [\cos(\beta-\beta') - \cos(\alpha-\alpha')]R(\beta')\chi(\alpha',\beta')\,d\alpha'. \quad (3.16)$$

One can verify, at this point, that χ can be written

$$\chi(\alpha,\beta) = e^{\epsilon^2 N\alpha}h(\beta). \quad (3.17)$$

We substitute this in (3.16) and obtain

$$h(\beta) = \frac{1}{4N}\int_{-\infty}^{\infty} \frac{\cos(\beta-\beta') + \epsilon^2 N\sin|\beta-\beta'|}{1+\epsilon^4N^2} e^{-\epsilon^2 N|\beta-\beta'|}R(\beta')h(\beta')\,d\beta'$$

$$\simeq \frac{1}{4N}\int_{-\infty}^{\infty} \cos(\beta-\beta')e^{-\epsilon^2 N|\beta-\beta'|}R(\beta')h(\beta')\,d\beta'. \quad (3.18)$$

We recapture (3.13) when R is a delta function by substituting $R(\beta') = \delta(\beta'-0)$ into (3.18), using the fact that our normalization requires that $h(0) = 1$, and noting that the eigenvalue, N, must have the value $N = \frac{1}{4}$.

More generally, since $h(\beta)$ must be even in β, (3.18) can be replaced by

$$h(\beta) = \frac{\cos\beta}{4N}\int_{-\infty}^{\infty} \cos\beta' e^{-\epsilon^2 N|\beta-\beta'|}R(\beta')h(\beta')\,d\beta' \quad (3.19)$$

and $h(\beta)$ can be written

$$h(\beta) = q(\beta)\cos\beta, \quad (3.20)$$

so that

$$q(\beta) = \frac{1}{8N}\int_{-\infty}^{\infty} (1+\cos 2\beta')e^{-\epsilon^2 N|\beta-\beta'|}R(\beta')q(\beta')\,d\beta'. \quad (3.21)$$

Whenever $R(\beta')$ is narrow compared to $(\epsilon^2 N)^{-1}$, we can anticipate that $q(\beta')$ will vary very slowly so that at $\beta = 0$ (since $q(0) = 1$)

$$1 \simeq \frac{1}{8N}\int_{-\infty}^{\infty} (1+\cos 2\beta')R(\beta')\,d\beta'. \quad (3.22)$$

Thus

$$N = \tfrac{1}{8}[\bar{R}(0) + \bar{R}(2)], \quad (3.23)$$

where $\bar{R}$ denotes the Fourier transform of R. That is, the growth rate depends only on the spectral power density of f at zero frequency and at that frequency ω which, when $f = \cos\omega t$, drives the parametric oscillator most effectively.

Now we can find the asymptotic behaviour of $q(\beta)$ by writing for (3.21)

$$q(\beta) \sim \frac{1}{8N}\int_{-\infty}^{\infty} (1+\cos 2\beta')\, R(\beta')\, e^{-\epsilon^2 N|\beta|}\, q(0)\, d\beta' = e^{-\epsilon^2 N|\beta|}. \tag{3.24}$$

Thus, for a broad range of ensembles, $f(t)$, the ensemble average of $u(t)u(\tau)$, displays a growth rate whose time constant is $(N\epsilon^2)^{-1}$ and a phase coherence which has the same time scale. The behaviour of $\langle u^2(t)\rangle$ implied by this, i.e.

$$\langle u^2(t)\rangle \sim \exp\{\tfrac{1}{4}\epsilon^2[\bar{R}(0)+\bar{R}(2)]\} \tag{3.25}$$

has been noted by Keller (1960) but, as far as I can tell, the results for $\langle u(t)u(\tau)\rangle$ given by (3.13) are new. In particular, his procedures do not permit the deductions of the next section which deals with ensembles $f(t)$ for which R is very broad.

It is useful to rewrite the results given in (3.13) in the normalized form

$$H(t-\tau) = \psi(t,\tau)/[\psi(t,t)\,\psi(\tau,\tau)]^{\frac{1}{2}} = e^{-\frac{1}{4}\epsilon^2|t-\tau|}\cos(t-\tau) \tag{3.26}$$

and to identify $\bar{H}(\omega)e^{\frac{1}{4}\epsilon^2(t+\tau)}$, where $\bar{H}(\omega)$ is the Fourier transform of $H(t-\tau)$, as the 'spectral power density at time $\frac{1}{2}(t+\tau)$'. That transform, $\bar{H}(\omega)$, is given by

$$\bar{H}(\omega) = \frac{\frac{1}{2}\epsilon^2}{(1-\omega)^2+\frac{1}{16}\epsilon^2}+\frac{\frac{1}{2}\epsilon^2}{(1+\omega)^2+\frac{1}{16}\epsilon^2}. \tag{3.27}$$

Thus, the 'spectral density of the response at time $\frac{1}{2}(t+\tau)$' is very strongly centred on unit frequency in accord with the arguments which led to the introduction of the hypotheses (3.7) and (3.8).

4. A deterministic limit

The most rapidly growing linear combination of the solutions of the Mathieu equation (Goldstein 1927)

$$u''+(1-\epsilon\cos 2t)\,u = 0, \tag{4.1}$$

has an amplitude which grows like $e^{\frac{1}{4}\epsilon t}$. It would be interesting to try to recover this result from the foregoing analysis and thereby test the breadth of applicability of that analysis. We can do this by choosing an ensemble of functions $f(t)$ (for (3.1)) each of which has the form

$$f(t) = \cos(2t+\mu), \tag{4.2}$$

where μ is a random variable distributed uniformly over the interval $0 < \mu \leqslant 2\pi$.

The auto-correlation of such an ensemble is

$$R(t-t') = \tfrac{1}{2}\cos[2(t-t')], \tag{4.3}$$

where, as in all of the ensembles discussed in this section, there is no advantage in normalizing R as we did in earlier sections.

Using (4.3) in the integral equation for $h(\beta)$, we obtain

$$h(\beta) = \frac{1}{8N}\int_{-\infty}^{\infty} \cos(\beta-\beta')\cos 2\beta'\, e^{-N\epsilon^2|\beta-\beta'|}\, h(\beta')\, d\beta'. \tag{4.4}$$

We note again that $h(\beta)$ must be even in β so that $\cos(\beta-\beta')$ can be replaced by $\cos\beta\cos\beta'$ and $h(\beta)$ can be written

$$h(\beta) = \cos\beta\, q(\beta).$$

We then obtain

$$q(\beta) = \frac{1}{32N}\int_{-\infty}^{\infty} (1+2\cos 2\beta'+\cos 4\beta')\, e^{-N\epsilon^2|\beta-\beta'|}\, q(\beta')\, d\beta'. \tag{4.5}$$

A broader class of functions $f(t)$ are those which form an 'almost deterministic problem', i.e. those for which

$$R(\beta') = \tfrac{1}{2}\cos 2\beta\, e^{-k\epsilon^2|\beta|} \tag{4.6}$$

and $k \ll 1/\epsilon$.

For this set, (4.5) becomes

$$q(\beta) = \frac{1}{32N}\int_{-\infty}^{\infty} (1+2\cos 2\beta'+ \cos 4\beta')\, e^{-N\epsilon^2|\beta-\beta'|-k\epsilon^2|\beta'|}\, q(\beta')\, d\beta', \tag{4.7}$$

and the Mathieu case is that for which $k = 0$.

It can be verified very easily when the analysis is finished that the discrepancies which arise when the cos $2\beta'$ and cos $4\beta'$ terms are omitted are very small indeed. The *a priori* argument merely notes that $q(\beta)$ may be one-signed and certainly is slowly varying; hence the oscillatory piece of the integrand cannot contribute much compared to that provided by the right-hand side of the following approximation to (4.7), i.e.

$$q(\beta) \simeq \frac{1}{32N}\int_{-\infty}^{\infty} e^{-N\epsilon^2|\beta-\beta'|-k\epsilon^2|\beta'|}\, q(\beta')\, d\beta'. \tag{4.8}$$

When we differentiate (4.8) twice with regard to β, we obtain

$$q'' = (N^2\epsilon^4 - \tfrac{1}{16}\epsilon^2 e^{-k\epsilon^2|\beta|})\, q. \tag{4.9}$$

Since q is even, we seek that solution of (4.9) for which $q'(0) = 0$ and we can confine our attention to the region $\beta > 0$. We can also adopt the new independent variable

$$x = k\epsilon^2(\beta-\beta_0),$$

where

$$\beta_0 = (k\epsilon^2)^{-1}\ln[(16N^2\epsilon^2)^{-1}].$$

Using this, we have

$$q_{xx} + (N^2/k^2)\,(e^{-x}-1)\, q = 0 \quad \text{in} \quad x > 0 \tag{4.10}$$

with

$$q_x(-k\epsilon^2\beta_0) = 0.$$

For the degenerate case, $k = 0$, we use (4.9) and note that, since we require that $q(\pm\infty) < M$, there is a continuous spectrum of eigenvalues in the range $-1/4\epsilon < N < 1/4\epsilon$. The largest of these corresponds precisely to the growth rate mentioned in the first paragraph of this section. The entire range corresponds to the range of growth rates of all possible solutions of our Mathieu equation.

For $k \neq 0$, the solution of (4.10) is

$$q = J_{2N/k}\left[\frac{e^{-\frac{1}{2}k\epsilon^2\beta}}{2k\epsilon}\right] \tag{4.11}$$

and this satisfies the boundary condition at $x = 0$ provided that

$$J'_{2N/k}\left(\frac{1}{2k\epsilon}\right) = 0. \tag{4.12}$$

For $k = O(1)$ and $N/k \gg 1$, we can use the asymptotic description of such Bessel functions to obtain instead of (4.12)

$$\frac{2N}{k} + \frac{Z_n}{2^{\frac{1}{3}}}\left(\frac{2N}{k}\right)^{\frac{1}{3}} \sim \frac{1}{2k\epsilon}, \tag{4.13}$$

where the numbers Z_n are the roots of the equation

$$A_i(-Z) = 0,$$

where $A_i(Z)$ denotes the Airy function.

The largest eigenvalue N corresponds to the root $Z_0 \simeq 1{\cdot}02$. When $k \simeq O(1)$, N stays close enough to $1/4\epsilon$ so that (4.13) is well approximated by

$$M_0(k) = |N| \sim \frac{1}{4\epsilon}\left(1 - \frac{Z_0}{2^{\frac{1}{3}}}(2k\epsilon)^{\frac{2}{3}}\right). \tag{4.14}$$

Since (4.10) implies two equal but opposite values of N corresponding to the same eigenfunction, the growth rate of a solution of our problem can be anything between $M_0(k)$ and $-M_0(k)$. That is, $\psi(t, \tau)$ is

$$\psi(t,\tau) \simeq [A\, e^{\epsilon^2 N(t+\tau)} + B\, e^{-\epsilon^2 N(t-\tau)}]\cos(t-\tau)\, q(t-\tau).$$

It is slightly disconcerting that (4.8) admits many eigenvalues and eigen-solutions. Much of this multiplicity may be an artefact of the approximation techniques, but some of it is necessarily a result of the lack of control (on the part of the investigator) as to whether the initial condition on each realization of $u(t)$ implies an initial phase which maximizes the early growth. In fact, the *continuum* of solutions of (4.8) for $k = 0$ reflects the fact that any growth rate $N \leqslant 1/4\epsilon$ can be realized by an appropriate choice for the distribution of initial phases.

On the other hand, it is clear that the results found in §3 for narrow R do not display a lack of uniqueness. This is consistent with the fact that the initial phase matters very little when f is so 'random' that the relation $\langle fu \rangle = 0$ is maintained not by a coherent 'orthogonality' between f and u (as in the Mathieu case), but by a statistically based, thorough lack of correlation. The absence of this coherent orthogonality renders the entering phase irrelevant for narrow R.

In accord with the foregoing, the proper interpretation of the results given in this paper for very broad R identifies them with the *maximum* growth rate on the *maximum* reflexion coefficient associated with a given ensemble of functions $f(t)$. For narrow R, no such interpretative restriction is implied!

5. Non-linear effects

As noted in §1, the differential equation which governs the lateral motion of an elastic string when non-linear effects are not ignored is

$$u''(t) + [1 + \tfrac{4}{3}u^2 - \epsilon f(t)]\, u = 0. \tag{5.1}$$

Appropriate boundary conditions are

$$u(0) = u_0 \ll 1, \quad u'(0) = 0,$$

but we will simplify the analysis again by moving the initial time, t_0, toward $-\infty$ with a corresponding decrease in $u(t_0)$.

When $f(t)$ is a known function of t, multi-scaling methods can be used (Cole 1968; Carrier & Pearson 1968) to establish the well-known fact that $u \simeq A(t_1)\cos t$ (with t_1 a slowly varying function of t) is a uniformly valid approximation to u. Furthermore, one can also verify that under such circumstances the role played by the non-linear term is duplicated when the $\frac{4}{3}u^3$ is replaced by A^2u. The success of this simplification (sometimes called the method of averaging) arises because the principal physical role of the u^3 term is to provide an augmentation of the time-averaged tension in the string. When $f(t)$ is $\cos 2t$, the (dimensionless) tension at very small amplitudes is unity and the system is so tuned that the oscillations grow. When the amplitude A has become large enough, the 'average tension', $1+A^2$, is no longer 'tuned' to $f(t)$ and, in particular, when $A^2 = \frac{1}{2}\epsilon$, no further growth will occur. Thus, the non-linear effects act primarily to detune the system and limit the amplitude of the oscillation.

In our problem, where $f(t)$ is described only stochastically, we must not only use the fact that, for each realization of f, $\frac{4}{3}u^3$ can be replaced by $\overline{u^2}u$ (where the bar denotes the time average over a cycle) but we must also use the somewhat cruder approximation wherein we take $\overline{u^2} \sim \langle u^2\rangle$. It is not easy to assess the accuracy of this approximation (I suspect it is very good); however, we are seeking here a delineation of the character of the non-linear effects and the accuracy of the numbers we get is not at issue.

All of these words underlie the replacement of (5.1) by

$$u''(t) + [1+\phi(t) - \epsilon f(t)]\,u(t) = 0, \tag{5.2}$$

where, in this section, again

$$\phi(t) = \langle u^2(t)\rangle. \tag{5.3}$$

It is convenient to introduce the variables

$$S = \int_0^t (1+\phi)^{\frac{1}{2}}\,dt \tag{5.4}$$

and

$$u = V(S)/(1+\phi)^{\frac{1}{4}}. \tag{5.5}$$

With these variables, (5.3) becomes

$$V'' + V - \frac{\epsilon f(t)\,V}{1+\phi} = \frac{3}{16}\frac{(\phi')^2 V}{(1+\phi)^2} + \frac{1}{4}\frac{\phi'' V}{(1+\phi)^2}. \tag{5.6}$$

Here ϕ' and ϕ'' denote $d\phi/dS$ and $d^2\phi/dS^2$.

Even when $f(t)$ has an extremely broad auto-correlation function, R, we expect ϕ to grow no faster than $e^{\frac{1}{2}\epsilon S}$; accordingly, we see that the right-hand side of (5.6) is *at most* of order ϵ^2 and we ignore those terms. Actually, we will carry out the details of the analysis here, only for $\langle f(t)f(t')\rangle = \delta(t-t')$ and, in that problem, the terms we ignore are only of order ϵ^4.

The pertinent solution of the expurgated (5.6) is

$$V = \cos S + \epsilon\int_0^S \sin(S-S')\frac{f(t')\,V(S')}{1+\phi(t')}\,dS'. \tag{5.7}$$

Following the arguments of §3, this can be replaced by

$$V(t) = \epsilon\int_{-\infty}^S \sin(S-S')\frac{f(t')\,V(S')}{1+\phi(t')}\,dS' \tag{5.8}$$

and $\langle V^2(S)\rangle = W(S)$ is given by

$$W(S) = \epsilon^2 \int_{-\infty}^{S}\int_{-\infty}^{S} \sin(S-S')\sin(S-S'')\frac{f(t')f(t'')\langle V(S')V(S'')\rangle}{[1+\phi(t')][1+\phi(t'')]}\,dS'\,dS''. \quad (5.9)$$

When $\langle f(t')f(t'')\rangle = \delta(t'-t'')$,
(5.9) becomes

$$W(S) = \epsilon^2 \int_{-\infty}^{S} \sin^2(S-S')\frac{W(S')}{[1+\phi(t')]^{\frac{3}{2}}}\,dS' \simeq \frac{\epsilon^2}{2}\int_{-\infty}^{S} W(S')[1+\phi(t')]^{-\frac{3}{2}}dS'. \quad (5.10)$$

The first equality arises because

$$\int \delta(t'-t'')\,dS'' = [1+\phi(t')]^{\frac{1}{2}};$$

the relative smallness of the integral associated with the part of $\sin^2(S-S')$ which is omitted in the second equality can be verified easily when the answer has been displayed.

When we differentiate (5.10), we get

$$W'(S) = \frac{\epsilon^2}{2}\frac{W(S)}{(1+\phi)^{\frac{3}{2}}},$$

i.e.

$$\frac{dW}{dt} = \frac{\epsilon^2}{2}\frac{W}{1+\phi}.$$

But $\phi = \langle u^2\rangle = W/(1+\phi)$,

so

$$\phi'(1+2\phi) = \tfrac{1}{2}\epsilon^2\phi \quad \text{and} \quad \ln\phi + 2\phi = \tfrac{1}{2}\epsilon^2 t. \quad (5.11)$$

Thus, when $\phi \ll 1$, $\phi \simeq e^{\frac{1}{2}\epsilon^2 t}$

and, when $\phi \gg 1$, $\phi \sim \tfrac{1}{4}\epsilon^2 t$.

Unless T_0, the tension in the string at rest, is much smaller than the yield stress times the cross-sectional area of the string, the equation will not be meaningful when $\phi \gg 1$. Even when this is the case, (5.11) is of interest in that it indicates that the amplitude-limiting effect of the detuning of the deterministic problem never occurs in this stochastic problem. The reason for this is not deeply hidden; the energy supply which supports the amplitude growth can be taken by the string from the input ($f(t)$) primarily in a spectral range which centres on twice the fundamental frequency of the string (see any discussion of Mathieu's equation). As the string detunes via the increasing tension in the $f = \cos 2t$ case, the available energy goes to zero. In the stochastic case, however, the available energy is spread over the entire spectrum (uniformly when $R = \delta(t-t')$) and no detuning occurs. The decrease in growth rate occurs because, with non-linear effects, the stored energy–amplitude relation does not permit an exponential amplitude growth.

It would be more complicated to deal with more general $f(t)$ than those of the foregoing analysis but, fortunately, it does not seem to be necessary. It is clear that, for narrow $R(t-t')$, the quantity $\tfrac{1}{2}\epsilon^2 t$ in (5.11) would be replaced by a close relative of

$$\tfrac{1}{4}\epsilon^2 t[\bar{R}(0) + \bar{R}(2[1+\phi])].$$

When f is almost deterministic, as in the examples of §4, $\bar{R}(\omega)$ is very small (but *not* zero) except near $\omega = 2$. Accordingly, the string will *almost* detune; that is, the early growth will resemble that of the linear deterministic problem but, at amplitudes where the growth would have ceased in the deterministic problem, the amplitude of the non-linear oscillation will continue to grow very slowly.

The work required to render these final paragraphs credibly quantitative seems not to be justified by any need.

6. The wave propagation problem

The mathematical problem associated with wave propagation across a region with irregular bottom topography differs from that associated with the foregoing linear string problem only in that the term $\cos x$ in (3.4) must be replaced by e^{ix}.

The procedure used in §2 must therefore be modified to take account of the fact that $u(x)$ is complex and to allow the calculation of $\langle |u'(x)|^2 \rangle$. The former requirement is met when we talk about $|u^2|$ instead of u^2 and the latter can be accomplished as follows.

Since
$$u(x) = e^{ix} + \epsilon \int_0^x \sin(x-x') f(x')\, u(x')\, dx', \tag{6.1}$$
it follows that
$$u'(x) = i\,e^{ix} + \epsilon \int_0^x \cos(x-x') f(x')\, u(x')\, dx' \tag{6.2}$$
and that
$$\langle |u'(x)|^2 \rangle = 1 + \epsilon^2 \int_0^x \int_0^x \cos(x-x') \cos(x-x'') R(x'-x'') \langle u(x')\, u^*(x'') \rangle\, dx'\, dx'', \tag{6.3}$$
where u^* denotes the complex conjugate of u.

The equation which corresponds to (3.9) is
$$u(x)\, u^*(y) = e^{i(x-y)} + \epsilon^2 \int_0^x dx' \int_0^y \sin(x-x') \sin(y-y') R(x'-y') \langle u(x')\, u^*(y') \rangle\, dy'. \tag{6.4}$$

It is evident that (6.4) can be solved by the procedures of §3 and that $\langle |u'(x)|^2 \rangle$ can then be calculated from (6.3). It also seems evident, without exhibiting details, that the discrepancies between the results for this problem and those for the string can only arise in connexion with the 'memory' of the phenomenon for its initial conditions. In our analysis, this memory was wiped out when we extended the initial-value co-ordinate (in §3) to $-\infty$. Thus, all details are necessarily repetitions of §3 and we omit them. We record only that
$$\tfrac{1}{2}\{\langle |u(L)|^2 \rangle + \langle |u'(L)|^2 \rangle\} = |\mathscr{R}|^2 + |I|^2 = e^{\frac{1}{4}\epsilon^2 L[\bar{R}(2) + \bar{R}(0)]}$$
except when R is so extremely wide (with $\geqslant O(1/\epsilon)$) that the integral equation, (3.21), must be treated with more care. In the oceanic context this is not the case and the reflexion coefficient must have the order of magnitude
$$|\mathscr{R}|^2 \simeq \frac{e^{\frac{1}{2}\epsilon^2 L} - 1}{e^{\frac{1}{2}\epsilon^2 L} + 1},$$

where L is the distance (2π times the number of wavelengths) traversed by the monochromatic wave. Since, during the deep water propagation, only the spectral region with wavelength greater than 50 miles is interesting, $\frac{1}{2}\epsilon^2 L \leqslant 0{\cdot}06$ and $|\mathscr{R}| \leqslant 0{\cdot}17$.

It pays to recall, at this stage, that the topography is *not* one-dimensional and that the one-dimensional treatment can be expected to overestimate the reflexion. Accordingly, we see that the cumulative reflexion associated with the irregular topography is weak. Note, however, that if the bottom topography were trigonometric, the reflexion of the spectral content of the wave at resonance (where the wave wavelength is twice the topographical wavelength) would exhibit a reflexion coefficient

$$|\mathscr{R}^2_{\text{trig}}| = \frac{e^{\frac{1}{2}\epsilon L}-1}{e^{\frac{1}{2}\epsilon L}+1},$$

which, with $\epsilon = 1/50$ and $L = 300$, gives

$$|\mathscr{R}_{\text{trig}}| \simeq 0{\cdot}95.$$

We conclude that very little of the tsunami energy in any spectral region is deterred by the *irregularities* of the deep water bottom topography from reaching any given 'target'.

7. Validity

Since all of the foregoing results depend on the validity of the hypotheses of (3.7) and (3.8), the credibility of this material should be enhanced by an assessment of the accuracy of those hypotheses. In this section we will indicate such an assessment for three sets of functions $f(x)$.

When $f(x)$ represents what is conventionally called Gaussian white noise, the quantities of interest are

$$\begin{aligned}\langle f(x)f(x')\rangle &= R(x-x') = \delta(x-x'),\\ \langle f(x)f(x')f(x'')f(x''')\rangle &= R(x-x')R(x''-x''')+R(x-x')R(x'-x''')\\ &\quad +R(x-x''')R(x'-x''),\\ \langle f(x)\ldots f(x^{(2n)})\rangle &= R(x-x')\,R(x''-x''')\ldots R(x^{(2n-1)}-x^{(2n)})\\ &\quad +\ldots([2n-1]\,[2n-3]\ldots[1] \text{ of them}),\end{aligned} \tag{7.1}$$

$$\langle f(x)\ldots f(x^{(2n-1)})\rangle = 0. \tag{7.2}$$

The solution of (3.1) (3.2) for any realization of $f(x)$ can be written in the form

$$u(x) = u_0(x)+\epsilon u_1(x)+\epsilon^2 u_2(x)+\ldots, \tag{7.3}$$

where

$$\begin{aligned}u_0(x) &= \cos x,\\ u_n(x) &= \int_0^x \sin(x-x')f(x')\,u_{n-1}(x')\,dx'\\ &= \int_0^x dx' \int_0^x \sin(x-x')\sin(x'-x'')f(x')f(x'')\,u_{n-2}(x'')\,dx''\\ &= \ldots.\end{aligned} \tag{7.4}$$

Using (7.1) and (7.2) in (7.4) we find

$$\langle u_n(x)\rangle = 0 \quad (n > 0),$$

and

$$\langle u_0^2(x)\rangle = \cos^2 x,$$

$$\langle u_0(x)\, u_1(x)\rangle = 0,$$

$$\langle u_1^2(x)\rangle = \int_0^x \sin(x-x')\, u_0^2(x)\, dx = \mathscr{L}[u_0^2(x)];$$

and, more generally,

$$\langle u_n(x)\, u_m(x)\rangle = 0 \quad (n \neq m), \tag{7.5}$$

$$\langle u_n^2(x)\rangle = \mathscr{L}[\langle u_{n-1}^2(x)\rangle]. \tag{7.6}$$

However, since

$$\langle u^2(x)\rangle = \langle u_0^2(x) + 2\epsilon u_0(x)\, u_1(x) + \epsilon^2[u_1^2(x) + 2u_0 u_2] + \ldots\rangle,$$

the foregoing results can be 'collected' in the form

$$\sum_{n=0}^{\infty} \epsilon^{2n}\langle u_n^2(x)\rangle = \cos^2 x + \epsilon^2 \sum_{n=0}^{\infty} \mathscr{L}[\epsilon^{2n}\langle u_n^2(x)\rangle],$$

i.e.

$$\langle u^2(x)\rangle = \cos^2 x = \epsilon^2 \mathscr{L}\langle u^2(x)\rangle. \tag{7.7}$$

This equation is identical with (3.10) and we have established that the result given by (3.12) is rigorously correct *for all values of* ϵ, when $f(x)$ is Gaussian white noise.

One can also use the recipes summarized under (7.4) to show directly that

$$\langle f(x)\, u(x)\rangle \equiv 0,$$

$$\langle f(x) f(x')\, u(x)\, u(x')\rangle \equiv \langle f(x) f(x')\rangle\, \langle u(x)\, u(x')\rangle.$$

We have already seen that the results obtained when $\epsilon \ll 1$ and when each $f(x)$ is trigonometric, i.e. where

$$f(x) = \cos(2x + \alpha),$$

are the same as those obtained by a direct deterministic treatment of the Mathieu equation. It is known that this deterministically obtained solution is uniformly valid in x and, therefore, we need only check the hypotheses (3.7), (3.8) to first order in ϵ. Omitting details, the fact is that, to this order, the hypotheses are satisfied.

Thus, for two very different extreme situations ($R(x-x')$ very narrow and $R(x-x')$ indefinitely wide) the hypotheses of this paper are completely valid.

It is interesting to check another, slightly broader family of functions $f(x)$. We again choose f to be associated with a Gaussian process but take

$$R(x-x') = 2k\, e^{-k|x-x'|}, \tag{7.8}$$

where $k > 0$.

Equations (7.1) and (7.2) are valid for this problem (but R is now given by (7.8)) and we can proceed as we did with the white noise problem. This time the integrals, all of which have exponential–trigonometric integrands, lead to a very messy set of recipes; the only interesting fact to be gleaned is that instead of getting (3.9), which has the form

$$\psi(t, \tau) = \cos t \cos \tau + \epsilon^2 \mathscr{L}[\psi(t, \tau)],$$

where $$\mathscr{L}[\psi] = \int_0^t\int_0^\tau \sin(t-t')\sin(\tau-\tau')R(t'-\tau')\,\psi(t,\tau')\,dt'\,d\tau',$$

we get $$\psi = \cos t\cos\tau + \epsilon^2\left\{\mathscr{L}[\psi] + O\left(\frac{\epsilon^2}{8k^2}\mathscr{L}[\psi]\right)\right\}.$$

Clearly, when $\epsilon^2/8k^2 \ll 1$, our hypotheses lead to an excellent estimate of the growth rate.

Since one cannot get firm results (other than loose inequalities) in problems where we don't know the statistics of f (other than $\langle f(x)f(x')\rangle$), the use of the truncation hypotheses used in this paper seems as useful a procedure for the explicit estimation of growth rates as are any which have yet been proposed.

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